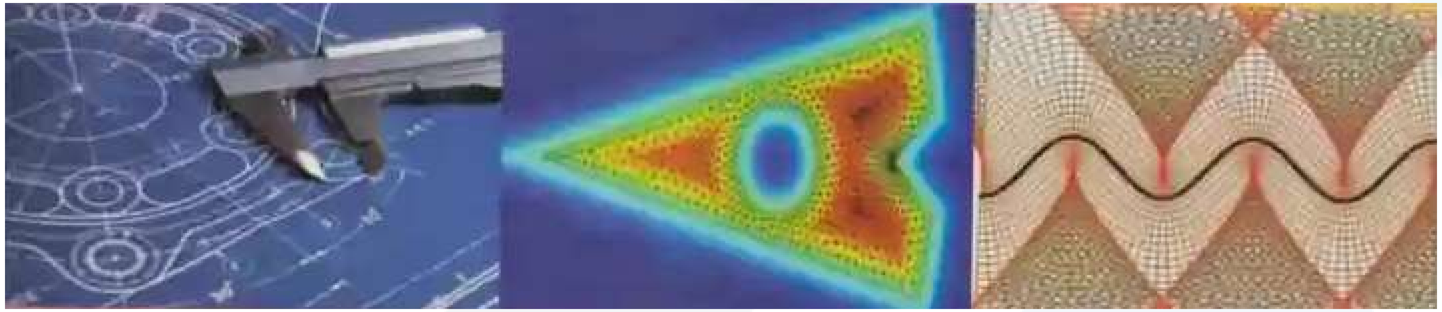
CEDC301: Mathematics Engineering

Exercises 3: Integration in the Complex Plan



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1. Evaluate the given integral

$$\oint_C \frac{\cos z}{z^3 - z^2}\,dz; \qquad C \text{ is the circle } |z| = \frac{1}{2}$$

$$\oint_C \frac{\cos z}{z^3 - z^2}\,dz = \oint_C \frac{\cos z}{z^2(z-1)}\,dz = \oint_{C_2} \frac{\frac{\cos z}{z-1}}{z^2}\,dz = \frac{2\pi i}{1!} f'(0)$$

$$f(z) = \frac{\cos z}{z-1} \Rightarrow f'(z) = \frac{\sin z - \cos z - z\sin z}{(z-1)^2}$$

$$\oint_C \frac{\cos z}{z^3 - z^2}\,dz = \frac{2\pi i}{1!}\left(\frac{-1}{1}\right) = -2\pi i$$

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$$\oint_C \frac{1}{2z^2+7z+3}\,dz; \quad C \text{ is the ellipse } \frac{x^2}{4}+y^2=1$$

$$\oint_C \frac{1}{2z^2+7z+3}\,dz = \oint_C \frac{1}{2(z+3)(z+1/2)}\,dz = \oint_{C_2} \frac{\frac{1}{2(z+3)}}{(z-(-1/2))}\,dz = 2\pi i f(-1/2)$$

$$f(z) = \frac{1}{2(z+3)} \Rightarrow \oint_C \frac{1}{2z^2+7z+3}\,dz = 2\pi i \frac{1}{5} = \frac{2\pi}{5} i$$

$$\oint_C \frac{e^{i\pi z}}{2z^2-5z+2}dz; \quad C \text{ is (a) } |z|=1, \text{ (b) } |z-3|=2, \text{ (c) } |z+3|=2$$

$$\oint_C \frac{e^{i\pi z}}{2z^2-5z+2}dz = \oint_C \frac{e^{i\pi z}}{2(z-2)(z-1/2)}dz = \oint_{C_2} \frac{\frac{1}{2(z+3)}}{(z-(-1/2))}dz = 2\pi i f(-1/2)$$

$$a.\ f(z) = \frac{e^{i\pi z}}{2(z-2)} \Rightarrow \oint_{C_2} \frac{\frac{e^{i\pi z}}{2(z-2)}}{z-1/2}dz = 2\pi i f(\frac{1}{2}) = 2\pi i(\frac{e^{i\pi/2}}{-3}) = \frac{2\pi}{3}$$

$$b.\ f(z) = \frac{e^{i\pi z}}{2(z-1/2)} \Rightarrow \oint_{C_2} \frac{\frac{e^{i\pi z}}{2z-1}}{z-2}dz = 2\pi i f(2) = 2\pi i(\frac{e^{2\pi i}}{3}) = \frac{2\pi}{3}i$$

$$c.\ \oint_C \frac{e^{i\pi z}}{2z^2-5z+2}dz = 0$$

2. Let $f(z) = z^n g(z)$, where n is a positive integer, $g(z)$ is entire, and $g(z) \neq 0$ for all z. Let C be a circle with center at the origin. Evaluate

$$\oint_C \frac{f'(z)}{f(z)}\,dz$$

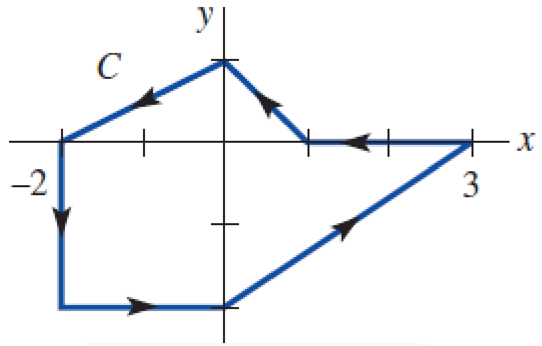
$$f(z) = z^n g(z) \Rightarrow f'(z) = z^n g'(z) + nz^{n-1} g(z)$$

$$\frac{f'(z)}{f(z)} = \frac{z^n g'(z) + nz^{n-1} g(z)}{z^n g(z)} = \frac{g'(z)}{g(z)} + \frac{n}{z}$$

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \oint_C \frac{g'(z)}{g(z)}\,dz + n\oint_C \frac{1}{z}\,dz = 0 + n(2\pi i) = 2n\pi i$$

3. Evaluate the given integral

$$\oint_C \frac{z}{z+i}\,dz; \quad C \text{ is the contour shown in figure}$$



Using the principle of deformation of contours we choose C to be a circular contour $|z+i| = ¼$.

$$z = -i + ¼ e^{it} \Rightarrow dz = ¼ i e^{it}$$

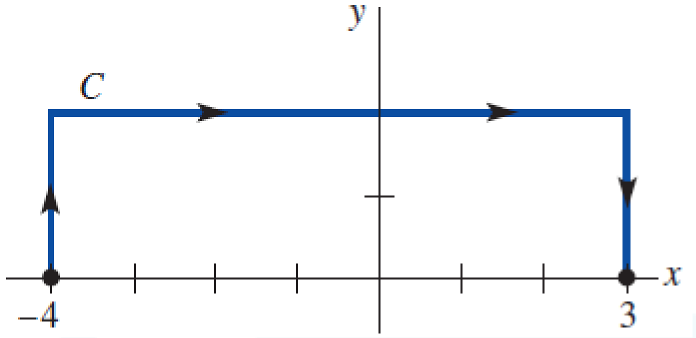
$$\oint_C \frac{z}{z+i}\,dz = i\int_0^{2\pi}\left(\frac{1}{4}e^{it} - i\right)dt = 2\pi$$

4. Evaluate the given integral

(a) $\int_C (x+iy)dz$

(b) $\int_C (x-iy)dz$;

C is the contour shown in figure



(a) $f(z) = z$ is entire, $\int_C (x+iy)dz$ is independent of the path C.

$$\int_C (x+iy)dz = \int_{-4}^{3} z dz = \left.\frac{z^2}{2}\right|_{-4}^{3} = -\frac{7}{2}$$

(b) $\int_C (x-iy)dz = \int_{C_1} (x-iy)dz + \int_{C_2} (x-iy)dz + \int_{C_3} (x-iy)dz$

On C_1, $x = 4$, $0 \le y \le 2$, $z = 4 + iy$, $dz = i\,dy$,

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$$\int_{C_1}(4-iy)idy = i\int_0^2(4-iy)dy = i\left(4y-\frac{i}{2}y^2\right)\Bigg|_0^2 = 2+8i$$

On C_2, $y = 2$, $-4 \le x \le 3$, $z = x + 2i$, $dz = dx$,

$$\int_{C_2}(x-2i)dx = \int_{-4}^3(x-2i)dx = \frac{1}{2}x^2 - 2ix\Big|_{-4}^3 = -\frac{7}{2}-14i$$

On C_3, $x = 3$, $0 \le y \le 2$, $z = 3 + iy$, $dz = idy$,

$$\int_{C_3}(3-iy)idy = i\int_2^3(x-2i)dx = \frac{1}{2}x^2 - 2ix\Big|_{-4}^3 = -\frac{7}{2}-14i$$

$$\int_C(x-iy)dz = 2+8i-\frac{7}{2}-14i-2-6i = -\frac{7}{2}-12i$$

5. Evaluate the given integral

$$\oint_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3} \right) dz; \quad |z| = 3$$

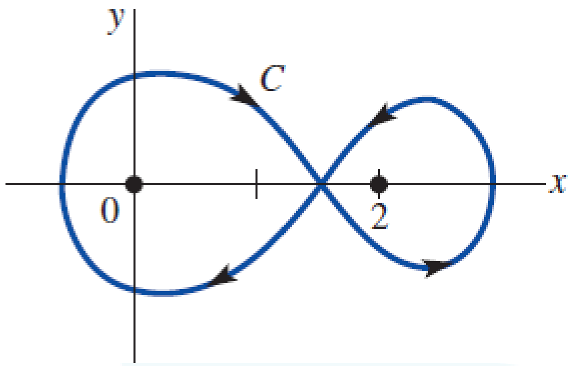
$$\oint_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3} \right) dz = \oint_C \frac{\cosh z}{(z-\pi)^3} dz - \oint_C \frac{\frac{1}{8}\sin^2 z}{\left(z-\frac{\pi}{2}\right)^3} dz$$

$$\oint_C \frac{\cosh z}{(z-\pi)^3} dz = 0 \qquad \oint_C \frac{\frac{1}{8}\sin^2 z}{\left(z-\frac{\pi}{2}\right)^3} dz = \frac{2\pi i}{2!}\left(-\frac{1}{4}\sin^2 \frac{\pi}{2} \right) = -\frac{\pi}{4} i$$

$$\oint_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3} \right) dz = \frac{\pi}{4} i$$

6. Evaluate the given integral

$$\oint_C \frac{3z+1}{z(z-2)^2}\,dz; \quad C \text{ is the contour shown in figure}$$



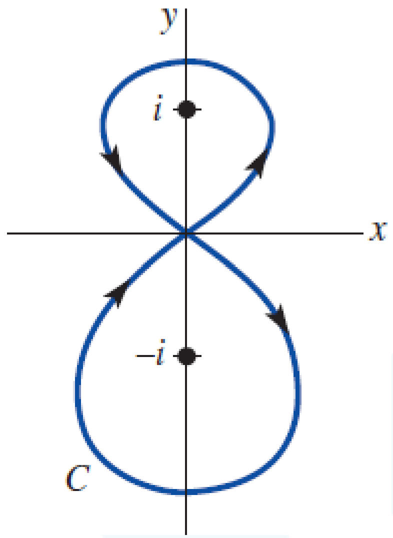
$$\oint_C \frac{3z+1}{z(z-2)^2}\,dz = \oint_{C_1} \frac{\frac{3z+1}{z}}{(z-2)^2}\,dz - \oint_{C_2} \frac{\frac{3z+1}{(z-2)^2}}{z}\,dz$$

$$\oint_{C_1} \frac{\frac{3z+1}{z}}{(z-2)^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i, \qquad \oint_{C_2} \frac{\frac{3z+1}{(z-2)^2}}{z}\,dz = 2\pi i\left(\frac{1}{4}\right) = \frac{\pi}{2}i$$

$$\oint_C \frac{3z+1}{z(z-2)^2}\,dz = -\frac{\pi}{2}i - \frac{\pi}{2}i = -\pi i$$

7. Evaluate the given integral

$$\oint_C \frac{e^{iz}}{(z+1)^2}\,dz; \quad C \text{ is the contour shown in figure}$$



$$\oint_C \frac{e^{iz}}{(z+1)^2}\,dz = \oint_{C_1} \frac{\frac{e^{iz}}{(z+i)^2}}{(z-i)^2}\,dz - \oint_{C_2} \frac{\frac{e^{iz}}{(z-i)^2}}{(z-(-i))^2}\,dz$$

$$\oint_{C_1} \frac{\frac{e^{iz}}{(z+i)^2}}{(z-i)^2}\,dz = \frac{2\pi i}{1!}\left(\frac{-4e^{-1}}{-8i}\right) = \pi e^{-1}, \qquad \oint_{C_2} \frac{\frac{e^{iz}}{(z-i)^2}}{(z-(-i))^2}\,dz = \frac{2\pi i}{1!}\left(\frac{0}{8i}\right) = 0$$

$$\oint_C \frac{e^{iz}}{(z+1)^2}\,dz = \pi e^{-1}$$